

The Exponential Integral

Part I - Derivation and Solution

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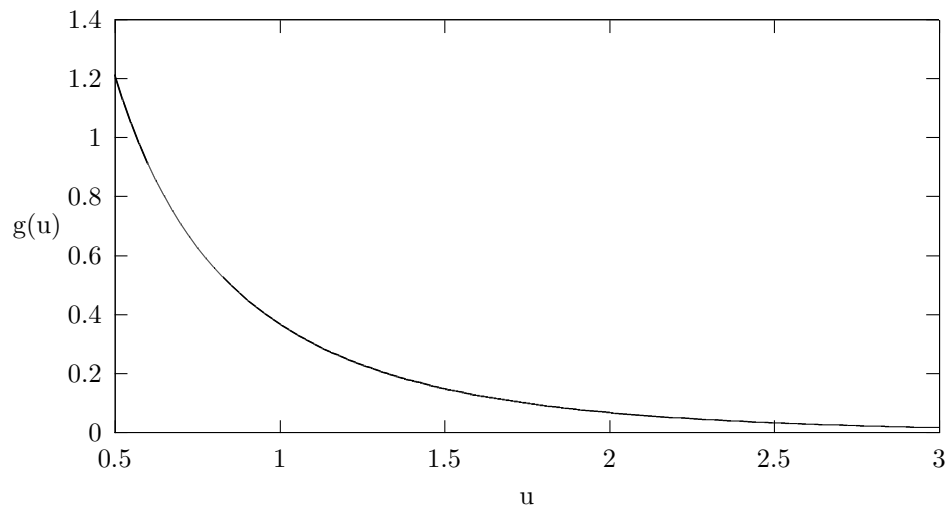
An exponential integral is defined as the definite integral of the ratio between an exponential function and its argument. This integral frequently arises in many fields of physics and engineering in general and quantum mechanics in particular. The exponential integral takes the following form...

$$f(a, b) = \int_a^b \text{Exp} \left\{ -u \right\} u^{-1} \delta u \dots \text{where} \dots 0 < a < b \quad (1)$$

The graph below plots the function...

$$g(u) = \text{Exp} \left\{ -u \right\} \times \frac{1}{u} \dots \text{where} \dots u = [0.50, 3.00] \quad (2)$$

Graph of the function g(u)



Our task will be to develop the mathematics of the exponential integral so as to solve our hypothetical problem below...

Our Hypothetical Problem

What is the value of the exponential integral over the interval [0.75, 2.50]?

Building The Mathematics

Note that we can rewrite the exponential function in the integrand of Equation (18) above as the following power series...

$$\text{Exp} \left\{ -u \right\} = \sum_{k=1}^{\infty} \frac{(-1)^k u^k}{k!} + 1 \quad (3)$$

Using Equation (3) above we can rewrite the integrand in Equation (18) above as...

$$\text{Exp} \left\{ -u \right\} u^{-1} = \left[\sum_{k=1}^{\infty} \frac{(-1)^k u^k}{k!} + 1 \right] u^{-1} = \sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k!} + u^{-1} \quad (4)$$

Using Equation (4) above we can rewrite Equation (18) above as...

$$f(a, b) = \int_a^b \left(\sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k!} + u^{-1} \right) \delta u = \int_a^b \sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k!} \delta u + \int_a^b u^{-1} \delta u \quad (5)$$

Note that we can rewrite Equation (5) above as...

$$f(a, b) = \sum_{k=1}^{\infty} \int_a^b \frac{(-1)^k u^{k-1}}{k!} \delta u + \int_a^b u^{-1} \delta u \quad (6)$$

Using Equation (6) above and Appendix Equations (19) and (21) below the solution to Equation (18) above is...

$$f(a, b) = \int_a^b \text{Exp} \left\{ -u \right\} u^{-1} \delta u = \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} (b^k - a^k) + \ln(b) - \ln(a) \quad (7)$$

Limit As Integral Upper Bound Goes To Infinity

We want to find the limit of Equation (7) above as b , which is the integral's upper bound, goes to infinity. Note that we can rewrite Equation (7) above as...

$$\begin{aligned} f(a, b) &= \int_a^b \text{Exp} \left\{ -u \right\} u^{-1} \delta u \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} (b^k - a^k) + \ln(b) - \ln(a) \\ &= - \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} a^k - \ln(a) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} b^k + \ln(b) \end{aligned} \quad (8)$$

We will start by determining the limit of the third term in Equation (8) above. It can shown that the limit of that power series is...

$$\lim_{b \rightarrow \infty} \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} b^k = -\ln(b) - \gamma \quad \dots \text{where} \dots \gamma = \text{Euler's Constant} \quad (9)$$

Note that the equation for Euler's Constant in Equation (9) above is...

$$\gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln(n) \approx 0.5772156649 \quad (10)$$

Using Equations (9) and (10) above we can rewrite Equation (8) above as...

$$g(a) = f(a, \infty) = \int_a^{\infty} \text{Exp} \left\{ -u \right\} u^{-1} \delta u = - \sum_{k=1}^{\infty} \frac{(-1)^k}{k k!} a^k - \ln(a) - \gamma \quad (11)$$

Note that the relationship between $f(a, b)$ and $g(a)$ above is...

$$f(a, b) = g(a) - g(b) \quad (12)$$

Author's Note: There are a number of approximations to the exponential integral of varying degrees of accuracy. The solution in Equation (11) above corresponds closely with the Gautschi solution but is much more intuitive as Gautschi's solution used a series of incomplete gamma functions rather than standard integration.

The Exponential Integral As The Limit Of The Incomplete Gamma Function

Using the Excel function GAMMALN note that the gamma function can be written as follows...

$$\Gamma(\alpha) = \int_0^{\infty} u^{\alpha-1} \text{Exp} \left\{ -u \right\} \delta u = \text{Exp} \left\{ \text{GAMMALN}(\alpha) \right\} \quad (13)$$

Using the Excel functions GAMMA.DIST and GAMMALN note that the lower incomplete gamma function can be written as follows...

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} \text{Exp} \left\{ -u \right\} \delta u = \text{GAMMA.DIST}(x, \alpha, 1, \text{true}) \times \text{EXP}(\text{GAMMALN}(\alpha)) \quad (14)$$

Note that the upper incomplete gamma function can be written as follows...

$$\Gamma(\alpha, x) = \int_x^{\infty} u^{\alpha-1} \text{Exp} \left\{ -u \right\} \delta u \quad (15)$$

The relationship between Equations (13), (14) and (15) above is...

$$\Gamma(\alpha, x) = \Gamma(\alpha) - \gamma(\alpha, x) \quad (16)$$

Using Excel functions we can rewrite Equation (16) above as...

$$\Gamma(\alpha, x) = \text{Exp} \left\{ \text{GAMMALN}(\alpha) \right\} \left(1 - \text{GAMMA.DIST}(x, \alpha, 1, \text{true}) \right) \quad (17)$$

The Point: Note that if the variable $\alpha = 0$ in Equation (15) above then that equation reduces to the exponential integral. The problem we have is that if we use Equation (16) to solve the exponential integral using built-in excel functions then the gamma and lower incomplete gamma functions are undefined when $\alpha = 0$. To get around this limitation what we want to do is define the exponential integral in terms of the following limit...

$$E_1(x) = \lim_{\alpha \rightarrow 0} \Gamma(\alpha, x) = \text{Exp} \left\{ \text{GAMMALN}(0.00000001) \right\} \left(1 - \text{GAMMA.DIST}(x, 0.00000001, 1, \text{true}) \right) \quad (18)$$

The Answer to Our Hypothetical Problem

Using the VBA code presented in **Appendix C** below the answer to our hypothetical problem is...

EXPINTU(0.75)	+	0.3403
EXPINTU(2.50)	-	0.0249
Answer	=	0.3154

Appendix

A. The solution to the first integral in Equation (6) above is...

$$\begin{aligned} \int_a^b \frac{(-1)^k u^{k-1}}{k!} \delta u &= \int_a^b u^{k-1} \times (-1)^k \times \frac{1}{k!} \times \delta u \\ &= \frac{(-1)^k}{k!} \int_a^b u^{k-1} \delta u \\ &= \frac{(-1)^k}{k!} \frac{u^k}{k} \Big|_a^b \\ &= \frac{(-1)^k b^k}{k k!} - \frac{(-1)^k a^k}{k k!} \\ &= \frac{(-1)^k}{k k!} (b^k - a^k) \end{aligned} \quad (19)$$

B. The solution to the second integral in Equation (6) above is...

$$\int_a^b u^{-1} \delta u = \ln(|x|) \Big|_a^b = \ln(|b|) - \ln(|a|) \quad (20)$$

When $u \geq 0$ then we can exclude the absolute value criteria from Equation (20) above so that equation becomes...

$$\int_a^b u^{-1} \delta u = \ln(b) - \ln(a) \quad \dots \text{when} \dots u \geq 0 \quad (21)$$

C. The VBA code for Equation (12) above is...

$$f(a, b) = g(a) - g(b) = \text{the following vba code} \dots \quad (22)$$

Name: EXPINTU

Purpose: Returns value of the [unbounded] exponential integral with lower bound u.

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Public Function EXPINTU(u As Double) As Double

 'Define calculation variables.

 Const EULER As Double = 0.577215664901533

 Const MAXIT As Integer = 100

 Const MINLOWERBOUND As Double = 0.000001

 Const MAXUPPERBOUND As Double = 10

 Const TOLERANCE As Double = 0.00000001

 Dim mValue As Double

 'Calculate return variable value.

 If u <= MINLOWERBOUND Then

 ElseIf u >= MAXUPPERBOUND Then

 mValue = 0

 Else

 'Calculate power series.

 Dim k As Integer

 Dim sum As Double

 Dim cvalue As Double

 Dim pvalue As Double

 sum = 0: pvalue = 0

 For k = 1 To MAXIT

 cvalue = (-u) ^ k * 1 / (k * FACTORIAL(k))

 sum = sum + cvalue

 If Abs(cvalue - pvalue) <= TOLERANCE Then Exit For

 pvalue = cvalue

 Next k

 'Define return variable value.

 mValue = -(EULER + LN(u) + sum)

 End If

 'Return value to caller.

 EXPINTU = mValue

End Function

'Name: LN

'Purpose: Returns natural log of [value].

Public Function LN(value) As Double

 LN = Application.WorksheetFunction.LN(value)

End Function

'Name: FACTORIAL

```
'Purpose: Returns factorial of [value].  
Public Function FACTORIAL(value) As Double  
    FACTORIAL = Application.WorksheetFunction.Fact(value)  
End Function
```